

From Infinitesimal Harmonic Transformations to Ricci Solitons

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Abstract

The concept of the Ricci soliton was introduced by Hamilton. Ricci soliton is defined by vector field and it's a natural generalization of Einstein metric. We have shown earlier that the vector field of Ricci soliton is an infinitesimal harmonic transformation. In our paper, we survey Ricci solitons geometry as an application of the theory of infinitesimal harmonic transformations.

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1. Harmonic diffeomorphisms and infinitesimal harmonic transformations

A smooth mapping $f : (M, g) \rightarrow (M', g')$ between two Riemannian manifolds is called *harmonic* (see [3]) if f provides an extremum of the Dirichlet functional $E_\Omega(f) = \frac{1}{2} \int_\Omega \|df\|^2 dV$ with respect to the variations of f that are compactly supported in a relatively compact open subset $\Omega \subset M$. (Here, dV is the volume element of the metric g .) The following theorem is true (see [3]).

Theorem 1.1. A smooth mapping $f : (M, g) \rightarrow (M', g')$ is harmonic if and only if it satisfies the Euler-Lagrange equations

$$g^{ij} (\partial_i \partial_j f^\beta - \Gamma_{ij}^k \partial_k f^\beta + \partial_i f^\beta \partial_j f^\gamma (\Gamma_{\beta\gamma}^\alpha \circ f)) = 0 \quad (1.1)$$

where $y^\alpha = f^\alpha(x^1, \dots, x^n)$ is local representation of f ; g^{ij} are local contravariant components of the metric tensor g ; Γ_{ij}^k and $\Gamma_{\beta\gamma}^\alpha$ are Christoffel symbols of (M, g) and (M', g') respectively; $i, j, k = 1, \dots, n = \dim M$ and $\alpha, \beta, \gamma = 1, \dots, n' = \dim M'$.

If we suppose that $\dim M = \dim M' = n$ and $f : (M, g) \rightarrow (M', g')$ is a diffeomorphism then f is locally represented by the following equations $y^i = x^i$ for $i, j, k, \dots = 1, 2, \dots, n$ and therefore the Euler-Lagrange equations (1.1) take the form

$$g^{ij} ((\Gamma_{ij}^k \circ f) - \Gamma_{ij}^k) = 0 \quad (1.2)$$

where Γ_{ij}^k and Γ'_{ij}^k are the Christoffel symbols of the Levi-Civita connection ∇ on (M, g) and ∇' on (M', g') respectively.

Suppose that we have a local one-parameter group of infinitesimal point transformations $f_t(x) = x'^k + t\xi^k$ generated by a vector field $\xi = \xi^k \partial_k$ on (M, g) for so-called canonical parameter t such that $t \in (-\xi, +\xi) \subset \mathbf{R}$. In this case the Lie derivative of the Christoffel symbols Γ_{ij}^k of the Levi-Civita connection ∇ has the form (see [21], pp. 8-9)

$$(L_\xi \Gamma_{ij}^k)t = \Gamma'_{ij}^k - \Gamma_{ij}^k = \nabla_i \nabla_j \xi^k - R_{ijl}^k \xi^l \quad (1.3)$$

where $\Gamma'_{ij}^k(x) = f_t^*(\Gamma_{ij}^k(x'))$.

Definition (see [14]; [19]). A vector field ξ on (M, g) is called an *infinitesimal harmonic transformation* if the one-parameter group of local transformations of (M, g) generated by ξ consists of local harmonic diffeomorphisms.

By the definition and (1.3) we deduce the following equation

$$\Delta\theta = 2\text{Ric}^*\xi \quad (1.4)$$

where ξ is an infinitesimal harmonic transformation and $\theta = g(\xi, \cdot)$ is its dual 1-form; $\Delta := dd^* + d^*d$ is the Hodge Laplacian on the space 1-forms $\Omega^1(M)$; Ric^* is the linear Ricci operator defined by the identity $g(\text{Ric}^*X, \cdot) = \text{Ric}(X, \cdot)$ for the tensor Ricci Ric and an arbitrary vector field X on M .

Theorem 1.2 (see [14], [19]). *The equation $\Delta\theta = 2\text{Ric}^*\xi$ is a necessary and sufficient condition for vector field ξ to be an infinitesimal harmonic transformation on a Riemannian manifold (M, g) .*

2. Examples of infinitesimal harmonic transformations

In this paragraph we will give five examples of infinitesimal harmonic transformations on Riemannian, nearly Kahlerian and Kahlerian manifolds.

Example 1. An infinitesimal isometric transformation on a Riemannian manifold is an infinitesimal harmonic transformation.

A vector field ξ on an n -dimensional Riemannian manifold (M, g) is an *infinitesimal isometric transformation* if $L_\xi g = 0$ where L_ξ is the Lie derivative in direction to ξ . By direct computation, we can deduce the following equalities $\Delta\theta = 2\text{Ric}^*\xi$ and $d^*\theta = 0$ for $\theta = g(\xi, \cdot)$. Moreover, these equalities are a necessary and sufficient condition for a vector field ξ to be an infinitesimal isometric transformation on a compact Riemannian manifold (M, g) (see [21], p. 221).

Example 2. An infinitesimal conformal transformation on a two-dimensional Riemannian manifold is a harmonic transformation.

Recall that a vector field ξ is an *infinitesimal conformal transformation* if $L_\xi g = -\frac{2}{n}(d^*\theta)g$ for $\theta = g(\xi, \cdot)$. By direct computation, we can deduce the following equality $\Delta\theta + (1 - \frac{2}{n})dd^*\theta = 2\text{Ric}^*\xi$. Moreover, by virtue of the Lihnerowicz theorem (see [12]) this equality is a necessary and sufficient condition for a vector field ξ to be an infinitesimal conformal transformation on a compact Riemannian manifold (M, g) . In particular, for $n = 2$ we have the equality $\Delta\theta = 2\text{Ric}^*\xi$. Therefore, any infinitesimal harmonic transformation

on a two-dimension compact Riemannian manifold is an infinitesimal conformal transformation.

Example 3 (see [19]). A holomorphic vector field on a nearly Kahlerian manifold is infinitesimal harmonic transformation.

Let the triplet (M, g, J) be a *nearly Kahlerian manifold* (see [6]) where $J \in T^*M \otimes TM$ such that $J^2 = -\text{id}_M$, $g(J, J) = g$ and $(\nabla_X J)Y + (\nabla_Y J)X = 0$ for any $X, Y \in TM$ and let ξ be a *holomorphic vector field* on (M, g, J) , i.e. $L_\xi J = 0$. In this case as we have proved in [19] that the following identity $\Delta\theta = 2\text{Ric}^*\xi$ holds.

Remark 4. On a compact Kahlerian manifold (M, g, J) , where as well known $\nabla J = 0$, a vector field ξ is holomorphic if and only if $\Delta\theta = 2\text{Ric}^*\xi$ (see [21], p. 280). Therefore, in particular, a vector field ξ on a compact Kahlerian manifold is an infinitesimal harmonic transformation if and only if ξ is holomorphic.

Example 5 (see [20]). A vector field ξ that makes a Riemannian metric g into a Ricci soliton metric is necessarily an infinitesimal harmonic transformation.

Let M be a smooth manifold. A *Ricci soliton* (g, ξ, λ) is a Riemannian metric g together with a vector field ξ on M and some constant λ that satisfies the equation $-2\text{Ric} = L_\xi g + 2\lambda g$ (see [1], pp 22-23).

The Lie derivative of ∇ has the following form (see [21], p. 52)

$$L_\xi \Gamma_{ij}^k = \frac{1}{2} g^{kl} (\nabla_i L_\xi g_{jl} + \nabla_j L_\xi g_{il} - \nabla_l L_\xi g_{ij}). \quad (2.1)$$

Substituting the identity $L_\xi g = -2(\text{Ric} + \lambda g)$ in (3.1) we find $L_\xi \Gamma_{ij}^k = g^{kl} (-\nabla_i R_{jl} - \nabla_j R_{il} + \nabla_l R_{ij})$ for local components R_{ij} of the Ricci tensor Ric . From the last equation we have $g^{ij} (L_\xi \Gamma_{ij}^k) = g^{kl} (-2\nabla_j R_l^j + \nabla_l s) = 0$ for the scalar curvature $s = g^{ij} R_{ij}$. Here we have taken advantage of the Schur's lemma $2\nabla_j R_l^j = \nabla_l s$.

Remark. If $\theta = dF$ for a smooth function $F : M \rightarrow \mathbf{R}$ then the equation of an infinitesimal harmonic transformation $\Delta\theta = 2\text{Ric}^*\xi$ can be written as $\Delta(\nabla_k F) = 2R_k^j \nabla_j F$ where $\Delta(\nabla_k F) = \nabla_k(\Delta F)$. On the other hand, if we put $\xi = \text{grad } F$ then from the equation of a Ricci soliton we conclude $\Delta F = s + n\lambda$ and hence the equation $\nabla_k(\Delta F) = 2R_k^j \nabla_j F$ is equal to $\nabla_k s = 2R_k^j \nabla_j F$. The last equation was proved by Hamilton for a gradient Ricci soliton (see [8]).

3. The Yano Laplacian

Let (M, g) be a compact Riemannian manifold. We may also assume that (M, g) is orientable; if (M, g) is not orientable, we have only to take an orientable twofold covering space of (M, g) . Denote by $S^p M$ the bundle of symmetric bilinear forms on (M, g) , δ^* the symmetric differentiation operator $\delta^* : C^\infty S^p M \rightarrow C^\infty S^{p+1} M$ and δ the linear differential operator $\delta : C^\infty S^{p+1} M \rightarrow C^\infty S^p M$ as the adjoint operator to δ^* with respect to the global scalar product on $S^p M$

$$\langle \varphi, \varphi' \rangle = \int_M \frac{1}{p!} g(\varphi, \varphi') dV,$$

which we get by integrating the pointwise inner product $g(\varphi, \varphi')$ for all $\varphi, \varphi' \in S^p M$.

Definition (see [17]; [18]). A differential operator $\square : C^\infty S^p M \rightarrow C^\infty S^p M$ is called the *Yano differential operator* if $\square = \delta\delta^* - \delta^*\delta$.

The Yano operator \square and Bochner Laplacian $\nabla^*\nabla$ are connected by the Weitzenbock formula $\square = \nabla^*\nabla + \mathfrak{R}_p$ for the symmetric endomorphism \mathfrak{R}_p of the bundle $S^p M$ such that \mathfrak{R}_p can be algebraically (even linearly) expressed through the curvature and Ricci tensors of (M, g) (see [17]; [18]). In particular for $p = 1$ we have $\mathfrak{R}_1 = -\text{Ric}^*$ and since $\square = \Delta - 2\text{Ric}^*$ (see [18]).

Remark. This form of the operator \square was used by K. Yano (see [22], p. 40) for the investigation of local isometric transformations of (M, g) . Therefore we have named \square as the Yano operator. Moreover, Yano has named a vector field ξ as *geodesic* if $\square\xi = 0$ (see [23]).

In view of told above we can formulate the following theorem.

Theorem 3.1. (see [19]). *A necessary and sufficient condition for a vector field ξ on a Riemannian manifold (M, g) to be infinitesimal harmonic transformation is that $\xi \in \text{Ker}\square$ for the Yano operator \square .*

From the following identity $\langle \square\varphi, \varphi' \rangle = \langle \varphi, \square\varphi' \rangle$, we conclude that \square is a self-adjoint differential operator (see [15]). In addition, the symbol σ of the Yano operator \square satisfies (see [17]) the following condition $\sigma(\square)(\vartheta, x)\varphi_x = -g(\vartheta, \vartheta)\varphi_x$ for an arbitrary $x \in M$ and $\vartheta \in T_x^* M - \{0\}$. Hence the Yano operator \square is the self-adjoint Laplacian operator and its kernel is a finite-dimensional vector space on compact (M, g) . In addition we recall that by virtue of the Fredholm alternative (see [16], p. 205) the vector spaces $\text{Ker}\square$ and $\text{Im}\square$ are orthogonal complement of each other with respect the global scalar product defined on compact (M, g) , i.e. $\Omega^p(M) = \text{Ker}\square \oplus \text{Im}\square$. In particular, for $p = 1$ we can formulate the following

Theorem 3.2. *The vector space $\text{Ker}\square$ of all infinitesimal harmonic transformations on compact Riemannian manifold (M, g) is a finite-dimensional vector space and the following orthogonal decomposition $\Omega^1(M) = \text{Ker}\square \oplus \text{Im}\square$ holds.*

For any conformal Killing vector field ζ and its dual 1-form ω on compact smooth manifold (M, g) we have $\langle \Delta\omega + (1 - 2n^{-1})d\delta^*\omega - 2\text{Ric}^*\zeta, \omega \rangle \geq 0$ (see [9]). From this inequality we conclude that $\langle \delta^*\omega, \delta^*\omega \rangle \geq 2n^{-1}\langle \delta\omega, \delta\omega \rangle \geq 0$ and hence $\langle \square\omega, \omega \rangle \geq 0$ for $n \geq 2$.

Remark. An infinitesimal harmonic transformation ξ is a harmonic vector field if and only if $\text{Ric}^*\xi = 0$. On a compact Riemannian manifold this infinitesimal harmonic transformation ξ must be a covariant constant vector field. Therefore, in particular, if Ricci tensor is a nonsingular tensor then determinant of the Ricci tensor is nonzero for every point $x \in M$ and in this case does not exist a nonzero harmonic vector field that belongs to the vector space $\text{Ker}\square$.

4. Three decomposition theorems

In this paragraph we will consider the vector space $\text{Ker}\square$ of all infinitesimal harmonic transformations on a compact Riemannian manifold. The following theorem is true.

Theorem 4.1. *If the vector field ξ is an infinitesimal harmonic transformation on a compact Riemannian manifold (M, g) then ξ is decomposed in the form $\xi = \xi' + \xi''$ where ξ' is an infinitesimal isometric transformation and ξ'' is an gradient infinitesimal harmonic transformation on (M, g) . This decomposition is necessarily orthogonal with respect to the global scalar product defined on (M, g) .*

Proof. The vector space $\text{Ker} \square \cap \text{Ker} d^*$ of all infinitesimal isometric transformations on a compact Riemannian manifold (M, g) is a subspace of the finite-dimensional vector space $\text{Ker} \square$ (see Exp. 1). On the other hand it is well known (see [16], p. 205) that by virtue of the Fredholm alternative vector spaces $\text{Im} d$ and $\text{Ker} d^*$ are orthogonal complement of each other with respect to the global scalar product on compact Riemannian manifold (M, g) , i.e. $\Omega^1(M) = \text{Ker} d^* \oplus \text{Im} d$. Therefore the vector space $\text{Ker} \square \cap \text{Ker} d$ of all infinitesimal gradient harmonic transformations must be an orthogonal complement of $\text{Ker} \square \cap \text{Ker} d^*$ with respect to the whole space $\text{Ker} \square$. This vector subspace consists of all gradient vector fields ∇F such that $\nabla_i(\Delta F) = 2R_i^j \nabla_j F$ for smooth scalar functions $F : M \rightarrow \mathbf{R}$.

Remark. The last result was known (see [23]) in the case of a compact Einstein n -dimensional ($n \geq 2$) manifold (M, g) with constant scalar curvature s .

Now we shell formulate the decomposition theorem of an arbitrary infinitesimal harmonic transformation on a compact Kahlerian manifold.

Theorem 4.2. *If ξ is a holomorphic vector field on a compact Kahlerian manifold (M, g, J) then ξ is decomposed in the form $\xi = \xi' + J\xi''$ where ξ' and ξ'' are both infinitesimal isometric transformations. This decomposition is necessarily orthogonal with respect to the global scalar product defined on (M, g, J) .*

Proof. On a compact Kahlerian manifold (M, g, J) , where as well known $\nabla J = 0$, a vector field ξ on a compact Kahlerian manifold is an infinitesimal harmonic transformation if and only if ξ is a holomorphic vector field (see Exp. 5). Therefore, by virtue of Theorem 4.1 we have the orthogonal decomposition $\xi = \xi' + \text{grad} F$ where ξ' is an infinitesimal isometric transformation and $\text{grad} F$ is a holomorphic vector field for some smooth scalar function F on (M, g) . On the other hand it is well known (see Theorem 6.8 of Chapter IV in [24]) that JX is an infinitesimal isometric transformation if a holomorphic vector field X is closed. Therefore we can state that $\xi = \xi' + \text{grad} F = \xi' + J\xi''$ where ξ'' is an infinitesimal isometric transformation.

Remark. Lihnerowicz has proved the following theorem (see [13]): A holomorphic vector field ξ on a compact Kahlerian manifold (M, g, J) with constant scalar curvature is decomposed in the form $\xi = \xi' + J\xi''$ where ξ' and ξ'' are both infinitesimal isometric transformation. Theorem 4.2 is a generalization of this theorem.

By virtue of the Fredholm alternative we shall prove the following theorem.

Theorem 4.3. *On a compact Riemannian manifold (M, g) of dimension ($n \geq 2$) with positive Ricci curvature an arbitrary infinitesimal conformal transformation ξ has the form $\xi = \xi' + \text{grad} F$ where ξ' is an infinitesimal isometric*

transformation and F is a some smooth scalar function on (M, g) such that the vector field $\text{grad}F$ is an infinitesimal conformal transformation. Moreover, if $L_{\text{grad}F}g = 0$ then manifold (M, g) is isometric to sphere \mathbf{S}^n in a Euclidian space \mathbf{R}^{n+1} .

Proof. The vector space of all infinitesimal conformal transformations on a compact Riemannian manifold (M, g) is a finite-dimensional vector space and the vector space of all infinitesimal isometric transformations is a subspace of this vector space. On the other hand, there does not exist a nonzero harmonic vector field on a compact Riemannian manifold with positive Ricci curvature (see Theorem 2.3 of Chapter II in [22]), then $\text{Im } d = \text{Ker } d$. Therefore on a compact Riemannian manifold (by virtue of the Fredholm alternative) for an arbitrary infinitesimal conformal transformation ξ the following decomposition is true $\xi = \xi' + \text{grad}F$ where ξ' is an infinitesimal isometric transformation and F is a some smooth scalar function on (M, g) such that the vector field $\xi'' = \text{grad}F$ with local coordinates $g^{ik}\nabla_k F$ is an infinitesimal conformal transformation. Then by direct computation, we obtain $L_{\xi}g = L_{\xi'}g + L_{\text{grad}F}g = L_{\text{grad}F}g = 2\nabla\nabla F$ and $\text{div } \xi = -\Delta F$. As a result we receive the following equality $L_{\text{grad}F}g = 2\nabla\nabla F = -(\Delta F)g$ from which we can conclude that the vector field $\xi'' = \text{grad}F$ is an infinitesimal conformal transformation also. It well known, if a compact Riemannian manifold (M, g) of dimension $n \geq 2$ admits a nonconstant scalar function F such that $\nabla\nabla F = n^{-1}(-\Delta F)g$ then (M, g) is conformal to a sphere \mathbf{S}^n in Euclidean space \mathbf{R}^{n+1} (see [11]). If in addition we suppose that $L_{\text{grad}F}g = 0$ then (M, g) must be isometric to a sphere \mathbf{S}^{n+1} (see [11]).

Remark. The vector space of all infinitesimal conformal transformations on (\mathbf{S}^n, \bar{g}) splits as the direct sum of the vector space of all infinitesimal isometric transformations and the vector space of gradient vector fields of first spherical harmonics of (\mathbf{S}^n, \bar{g}) . In particular, the vector space of all infinitesimal conformal transformations on (\mathbf{S}^2, \bar{g}) has dimension equal to 6 and admits decomposition in the sum of two subspaces (see [4]). Three of the dimensions arise from $\bar{\nabla}F$ where F is a spherical harmonic. The other three dimensions come from the infinitesimal isometric transformations for the standard metric \bar{g} on \mathbf{S}^2 . Therefore our decomposition the vector space of infinitesimal conformal transformations on a compact Riemannian manifold is an analog of above decomposition on a sphere (\mathbf{S}^n, \bar{g}) .

5. Ricci solitons

Let (g, ξ, λ) be a *Ricci soliton* on a smooth n -dimensional manifold M (see Exp. 4), where g is Riemannian metric and ξ is a smooth vector field on M such that the identity

$$-2\text{Ric} = L_{\xi}g + 2\lambda g \quad (5.1)$$

holds for some constant λ (see [1], p. 22; [2], p. 353). Ricci soliton is called *steady*, if $\lambda = 0$, *shrinking*, if $\lambda < 0$, and, finally, *expanding*, if $\lambda > 0$.

In case $\xi = \text{grad}F$ for some smooth function $F : M \rightarrow \mathbf{R}$ the equation can be rewritten as

$$-Ric = \nabla \nabla F + \lambda g. \quad (5.2)$$

and (g, ξ, λ) is called a *gradient Ricci soliton* (see [1], p. 22; [2], p. 353). Moreover, (M, g) is called a *trivial Ricci soliton*, if $F = \text{const}$ and hence (M, g) is an Einstein manifold.

By Example 4 a vector field ξ that makes a Riemannian metric g into a metric of a Ricci soliton is necessarily an infinitesimal harmonic transformation. In addition, by the first decomposition theorem a harmonic transformation ξ on a compact Riemannian manifold (M, g) has the form $\xi = \xi' + \xi''$ where ξ' is an infinitesimal isometric transformation and ξ'' is a gradient infinitesimal harmonic transformation on (M, g) . By these propositions we can rewrite the identity (5.1) as

$$-2Ric = L_\xi g + 2\lambda g = L_{\xi' + \xi''} g + 2\lambda g = 2\nabla \nabla F + 2\lambda g$$

where $\xi'' = \text{grad}F$ for some smooth scalar function F . Now we can formulate the following proposition.

Theorem 5.1. *The vector field ξ of any Ricci soliton (g, ξ, λ) on a compact smooth manifold M has the form $\xi = \xi' + \xi''$ where ξ' is an infinitesimal isometric transformation and ξ'' is a gradient infinitesimal harmonic transformation and therefore (g, ξ, λ) is a gradient Ricci soliton.*

Remark. By means of Perelman work [15] and previous others, see Hamilton [7] for dimension two and Ivey [10] for dimension 3 we know that an every compact Ricci soliton is a gradient Ricci soliton. And hence the Perelman-Hamilton-Ivey propositions is a corollary of our theorem about infinitesimal harmonic transformations on a compact smooth manifold.

Now we take the divergence of the Ricci tensor of (g, ξ, λ) . By using the equation (5.2) we have $\text{div} Ric = g^{ij} \nabla_i R_{jk} = \frac{1}{2} \nabla_k s = -g^{ij} \nabla_i \nabla_j \nabla_k F = \Delta(\nabla_k F)$. Using this equation and the Schur's Lemma $\text{div} Ric = 2ds$ we get

$$ds = 2d(\Delta F). \quad (5.3)$$

Then by means of the equation (5.3) we have

$$\langle \xi, ds \rangle = 2\langle \xi, d(\Delta F) \rangle = 2\langle d^* \xi, \Delta F \rangle = 2\langle d^* dF, \Delta F \rangle = 2\langle \Delta F, \Delta F \rangle$$

that is equivalent to $\int_M \xi(s) dV = \int_M (\Delta F)^2 dv \geq 0$. By means of this inequity we can formulate the following proposition (see [20])

Theorem 5.2. *If a shirking Ricci soliton (g, ξ, λ) on compact smooth manifold M satisfies the condition $L_\xi s \leq 0$ then this soliton is trivial.*

Remark. It is well known that a compact steady or expanding Ricci soliton (g, ξ, λ) is a gradient soliton (see [15]) and on the other hand a compact gradient steady or expanding Ricci soliton is a trivial soliton (see [8]). On the other hand every shirking compact Ricci soliton when $n > 3$ and the Weyl tensor is zero is trivial (see [5] and [25]). But there is the open problem (see [5], p. 11): Are the special conditions in dimension $n \geq 4$ assuring that a shirking compact Ricci

soliton is trivial? Our Theorem 5.2 may be is one of possible answers to this question.

We have proved (see [17]) that on a compact Riemannian manifold (M, g) does not exist an infinitesimal harmonic transformation ξ such that $\text{Ric}(\xi, \xi) < 0$. And in addition if $\text{Ric}(\xi, \xi) \leq 0$ for an infinitesimal harmonic transformation ξ such that $\xi \neq 0$ then ξ is a parallel vector field.

On the other hand, the vector field ξ that makes a Riemannian metric g into a metric of Ricci soliton must be a null-vector field if $\nabla \xi = 0$. These two facts can be used to formulate the following assertion (see [20]).

Corollary. *A Riemannian metric g on a compact smooth manifold M can not be metric of a Ricci soliton (g, ξ, λ) if $\text{Ric}(\xi, \xi) < 0$. If $\text{Ric}(\xi, \xi) \leq 0$ then (g, ξ, λ) is a trivial Ricci soliton.*

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